

NOTES ON FORMAL DEFORMATIONS OF HOM-ASSOCIATIVE AND HOM-LIE ALGEBRAS

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ABSTRACT. The aim of this paper is to extend to Hom-algebra structures the theory of formal deformations of algebras which was introduced by Gerstenhaber for associative algebras and extended to Lie algebras by Nijenhuis-Richardson. We deal with Hom-associative and Hom-Lie algebras. We construct the first groups of a deformation cohomology and give several examples of deformations. We provide families of Hom-Lie algebras deforming Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ and describe as formal deformations the q -deformed Witt algebra and Jackson $\mathfrak{sl}_2(\mathbb{K})$.

INTRODUCTION

Discretization of vector fields via twisted derivations leads to Hom-Lie and quasi-Hom-Lie structures. This quasi-deformation method was devised in [12, 15, 16]. We have introduced in [22], the structure of Hom-associative algebras generalizing associative algebras to a situation where associativity law is twisted by a linear space homomorphism. This provided a different way for constructing Hom-Lie algebras, introduced in [12], by extending the fundamental construction of Lie algebras from associative algebras via commutator bracket multiplication. We showed that the commutator product defined using the multiplication in a Hom-associative algebra leads naturally to Hom-Lie algebras. The notion, constructions and properties of the enveloping algebras of Hom-Lie algebras are yet to be properly studied in full generality. An important progress in this direction has been made recently by D. Yau [25].

The idea to deform algebraic, analytic and geometric structures within the appropriate category is obviously not new. The first modern appearance is often attributed to Kodaira and Spencer and deformations in connection to complex structures on complex manifolds. This was, however, soon extended and generalized in an algebraic-homological setting by Gerstenhaber, Grothendieck and Schlessinger. Nowadays deformation-theoretic ideas penetrate most aspects of both mathematics and physics and cut to the core of theoretical and computational problems. In the case of Lie algebras, quantum deformations (or q -deformations) and quantum groups associated to Lie algebras has been investigated for over twenty years, still growing richer by the minute. This area began a period of rapid expansion around 1985 when Drinfel'd and Jimbo independently considered deformations of $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of a Lie algebra \mathfrak{g} , motivated, among other things, by their applications to the Yang-Baxter equation and quantum inverse scattering methods. The formal deformation theory was introduced in 1964 by Gerstenhaber [11] for associative algebras, and in 1967 by Nijenhuis and Richardson for Lie algebras [24]. In this theory the scalar's field is extended to the power series ring. The fundamental results of Gerstenhaber's theory connect deformation theory with the suitable cohomology groups. There is no general deformation cohomology theory. Other approaches to study deformation exist [2, 4, 5, 6, 8, 9, 18], see [21] for a review. In this paper we aim to extend the formal deformation theory to Hom-associative and Hom-Lie algebras and start to construct a cohomology theory adapted to this deformation theory. In Section 1, we recall the basic definitions of Hom-algebra structures and introduce a definition of Hom-Poisson algebra which will play an interesting role in the deformation theory of commutative Hom-associative algebras. In

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Section 2, we introduce formal deformations for Hom-associative and Hom-Lie algebras. Since the Hom-Lie algebras enlarge the category of Lie algebras, then one may expect non-trivial deformations for rigid associative and Lie algebras. Indeed, we provide Hom-Lie deformations of the classical Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ and show that the Jackson $\mathfrak{sl}_2(\mathbb{K})$ is also a Hom-Lie formal deformation of $\mathfrak{sl}_2(\mathbb{K})$. In Section 3, we give some elements of cohomology of Hom-associative algebras which are used in Section 4 to describe deformations of Hom-associative algebras in terms of cohomology. Section 5 is dedicated to cohomology and deformations of Hom-Lie algebras. In Section 6, we provide families of Hom-Lie algebras deforming Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ and describe as formal deformations the q -deformed Witt algebra.

1. HOM-ALGEBRA STRUCTURES AND MODULES

A Hom-algebra structure is a multiplication on a vector space where the structure is twisted by a linear space homomorphism. In the following we summarize the definitions of Hom-associative, Hom-Leibniz and Hom-Lie algebraic structures [22] generalizing the well known associative, Leibniz and Lie-admissible algebras. Also we introduce notion of Hom-Poisson algebras generalizing Poisson algebras to the Hom-Lie context, and define the notion of modules over Hom-associative algebras.

1.1. Definitions. Throughout the article we let \mathbb{K} be an algebraically closed field of characteristic 0 and V be a vector space over \mathbb{K} .

Definition 1.1. A *Hom-associative algebra* over V is a triple (V, μ, α) where $\mu : V \times V \rightarrow V$ is a bilinear map and $\alpha : V \rightarrow V$ is a linear map, satisfying

$$(1.1) \quad \mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)).$$

In the language of Hopf algebra a *Hom-associative algebra* over V is a linear map $\mu : V \otimes V \rightarrow V$ and a linear map α satisfying

$$(1.2) \quad \mu(\alpha(x) \otimes \mu(y \otimes z)) = \mu(\mu(x \otimes y) \otimes \alpha(z)).$$

The tensor product of Hom-associative algebras (V_1, μ_1, α_1) and (V_2, μ_2, α_2) is defined in an obvious way as the Hom-associative algebra $(V_1 \otimes V_2, \mu_1 \otimes \mu_2, \alpha_1 \otimes \alpha_2)$.

A linear map $\phi : V \rightarrow V'$ is a morphism of Hom-associative algebras if

$$\mu' \circ (\phi \otimes \phi) = \phi \circ \mu \quad \text{and} \quad \phi \circ \alpha = \alpha' \circ \phi.$$

In particular, Hom-associative algebras (V, μ, α) and (V, μ', α') are isomorphic if there exists a bijective linear map ϕ such that

$$\mu = \phi^{-1} \circ \mu' \circ (\phi \otimes \phi) \quad \text{and} \quad \alpha = \phi^{-1} \circ \alpha' \circ \phi.$$

The Hom-associative algebra is called unital if it admits a unity, i.e. an element $e \in V$ such that $\mu(x, e) = \mu(e, x) = x$ for all $x \in V$. The unity may be also expressed by a linear map $\eta : \mathbb{K} \rightarrow V$ defined by $\eta(c) = ce$ for all $c \in \mathbb{K}$. Let (V, μ, α, η) and $(V', \mu', \alpha', \eta')$ be two unital Hom-associative algebras. Then morphisms of unital Hom-associative algebras are required also to preserve the unital structure, i.e. satisfy $f \circ \eta = \eta'$, and in the definition of isomorphism of unital Hom-associative algebras it is also required that $\eta = f^{-1} \circ \eta'$.

The Hom-Lie algebras were introduced by Hartwig, Larsson and Silvestrov in [12] motivated initially by examples of deformed Lie algebras coming from twisted discretizations of vector fields.

Definition 1.2. A *Hom-Lie algebra* is a triple $(V, [\cdot, \cdot], \alpha)$ consisting of a linear space V , bilinear map $[\cdot, \cdot] : V \times V \rightarrow V$ and a linear space homomorphism $\alpha : V \rightarrow V$ satisfying

$$\begin{aligned} [x, y] &= -[y, x] \quad (\text{skew-symmetry}) \\ \bigcirc_{x,y,z} [\alpha(x), [y, z]] &= 0 \quad (\text{Hom-Jacobi condition}) \end{aligned}$$

for all x, y, z from V , where $\bigcirc_{x,y,z}$ denotes summation over the cyclic permutation on x, y, z .

In a similar way we have the following definition of Hom-Leibniz algebra.

Definition 1.3. A *Hom-Leibniz algebra* is a triple $(V, [\cdot, \cdot], \alpha)$ consisting of a linear space V , bilinear map $[\cdot, \cdot] : V \times V \rightarrow V$ and a homomorphism $\alpha : V \rightarrow V$ satisfying

$$(1.3) \quad [[x, y], \alpha(z)] = [[x, z], \alpha(y)] + [\alpha(x), [y, z]].$$

Note that if a Hom-Leibniz algebra is skewsymmetric then it is a Hom-Lie algebra.

We introduce in the following the definition of Hom-Poisson structure which involves naturally in the deformation theory.

Definition 1.4. A *Hom-Poisson algebra* is a quadruple $(V, \mu, \{\cdot, \cdot\}, \alpha)$ consisting of a linear space V , bilinear maps $\mu : V \times V \rightarrow V$ and $\{\cdot, \cdot\} : V \times V \rightarrow V$, and a linear space homomorphism $\alpha : V \rightarrow V$ satisfying

- (1) (V, μ, α) is a commutative Hom-associative algebra,
- (2) $(V, \{\cdot, \cdot\}, \alpha)$ is a Hom-Lie algebra,
- (3) for all x, y, z in V ,

$$(1.4) \quad \{\alpha(x), \mu(y, z)\} = \mu(\alpha(y), \{x, z\}) + \mu(\alpha(z), \{x, y\}).$$

The condition (1.4) expresses the compatibility between the multiplication and the Poisson bracket. It can be reformulated equivalently as

$$(1.5) \quad \{\mu(x, y), \alpha(z)\} = \mu(\{x, z\}, \alpha(y)) + \mu(\alpha(x), \{y, z\})$$

for all x, y, z in V . Note that in this form it means that $ad_z(\cdot) = \{\cdot, z\}$ is a sort generalization of derivation of associative algebra defined by μ , and also it resembles the identity (1.3) in the definition for Leibniz algebra.

We also recall in the following the structure of module over Hom-associative algebras.

Definition 1.5. Let $\mathcal{A} = (V, \mu, \alpha)$ be a Hom-associative \mathbb{K} -algebra. An \mathcal{A} -module (left) is a triple (M, f, γ) where M is \mathbb{K} -vector space and f, γ are \mathbb{K} -linear maps, $f : M \rightarrow M$ and $\gamma : V \otimes M \rightarrow M$, such that the following diagram commute:

$$\begin{array}{ccc} V \otimes V \otimes M & \xrightarrow{\mu \otimes f} & V \otimes M \\ \downarrow \alpha \otimes \gamma & & \downarrow \gamma \\ V \otimes M & \xrightarrow{\gamma} & M \end{array}$$

Remark 1.6. A Hom-associative \mathbb{K} -algebra $\mathcal{A} = (V, \mu, \alpha)$ is a left \mathcal{A} -module with $M = V$, $f = \alpha$ and $\gamma = \mu$.

2. FORMAL DEFORMATIONS OF HOM-ASSOCIATIVE ALGEBRAS AND HOM-LIE ALGEBRAS

In this section we extend to Hom-algebra structures the formal deformation theory introduced by Gerstenhaber for associative algebras [11], and by Nijenhuis and Richardson for Lie algebras [24]. More precisely, we define the concept of deformation for Hom-associative algebras and Hom-Lie algebras and define the suitable 1^{st} and 2^{nd} cohomology groups adapted to formal deformation.

Let V be a \mathbb{K} -vector space and $\mathcal{A}_0 = (V, \mu_0, \alpha_0)$ be a Hom-associative algebra and $L_0 = (V, [\cdot, \cdot]_0, \alpha_0)$ be a Hom-Lie algebra. Let $\mathbb{K}[[t]]$ be the power series ring in one variable t and coefficients in \mathbb{K} and $V[[t]]$ be the set of formal power series whose coefficients are elements of V , ($V[[t]]$ is obtained by extending the coefficients domain of V from \mathbb{K} to $\mathbb{K}[[t]]$). Then $V[[t]]$ is a $\mathbb{K}[[t]]$ -module. When V is finite-dimensional, we have $V[[t]] = V \otimes_{\mathbb{K}} \mathbb{K}[[t]]$. Note that V is a submodule of $V[[t]]$. Given a \mathbb{K} -bilinear map $f : V \times V \rightarrow V$, it admits naturally an extension to a $\mathbb{K}[[t]]$ -bilinear map $f : V[[t]] \times V[[t]] \rightarrow V[[t]]$, that is, if $x = \sum_{i \geq 0} a_i t^i$ and $y = \sum_{j \geq 0} b_j t^j$ then $f(x, y) = \sum_{i \geq 0, j \geq 0} t^{i+j} f(a_i, b_j)$. The same holds for linear maps.

Definition 2.1. Let V be a \mathbb{K} -vector space and $\mathcal{A}_0 = (V, \mu_0, \alpha_0)$ be a Hom-associative algebra. A *formal Hom-associative deformation* of \mathcal{A}_0 is given by the $\mathbb{K}[[t]]$ -bilinear and $\mathbb{K}[[t]]$ -linear maps $\mu_t : V[[t]] \times V[[t]] \rightarrow V[[t]]$ and $\alpha_t : V[[t]] \rightarrow V[[t]]$ of the form

$$\mu_t = \sum_{i \geq 0} \mu_i t^i, \quad \text{and} \quad \alpha_t = \sum_{i \geq 0} \alpha_i t^i$$

where each μ_i is a \mathbb{K} -bilinear map $\mu_i : V \times V \rightarrow V$ (extended to be $\mathbb{K}[[t]]$ -bilinear) and each α_i is a \mathbb{K} -linear map $\alpha_i : V \rightarrow V$ (extended to be $\mathbb{K}[[t]]$ -linear), such that holds for $x, y, z \in V$ the following formal Hom-associativity condition:

$$(2.1) \quad \mu_t(\alpha_t(x), \mu_t(y, z)) - \mu_t(\mu_t(x, y), \alpha_t(z)) = 0.$$

Definition 2.2. Let V be a \mathbb{K} -vector space and $L_0 = (V, [\cdot, \cdot]_0, \alpha_0)$ be a Hom-Lie algebra. A *formal Hom-Lie deformation* of L_0 is given by the $\mathbb{K}[[t]]$ -bilinear and $\mathbb{K}[[t]]$ -linear maps $[\cdot, \cdot]_t : V[[t]] \times V[[t]] \rightarrow V[[t]]$, $\alpha_t : V[[t]] \rightarrow V[[t]]$ of the form

$$[\cdot, \cdot]_t = \sum_{i \geq 0} [\cdot, \cdot]_i t^i, \quad \text{and} \quad \alpha_t = \sum_{i \geq 0} \alpha_i t^i$$

where each $[\cdot, \cdot]_i$ is a \mathbb{K} -bilinear map $[\cdot, \cdot]_i : V \times V \rightarrow V$ (extended to be $\mathbb{K}[[t]]$ -bilinear) and each α_i is a \mathbb{K} -linear map $\alpha_i : V \rightarrow V$ (extended to be $\mathbb{K}[[t]]$ -linear), and satisfying for $x, y, z \in V$ the following conditions:

$$(2.2) \quad \begin{aligned} [x, y]_t &= -[y, x]_t \quad (\text{skew-symmetry}) \\ \odot_{x, y, z} [[\alpha_t(x), [y, z]_t]_t &= 0. \quad (\text{Hom-Jacobi condition}) \end{aligned}$$

Remark 2.3. The skew-symmetry of $[\cdot, \cdot]_t$ is equivalent to the skew-symmetry of all $[\cdot, \cdot]_i$ for $i \in \mathbb{Z}_{\geq 0}$.

We call the condition (2.1) (respectively (2.2)) deformation equation of Hom-associative (respectively Hom-Lie) algebra.

Example 2.4 (Jackson $\mathfrak{sl}_2(\mathbb{K})$). In this example, we will consider the Hom-Lie algebra Jackson $\mathfrak{sl}_2(\mathbb{K})$ which is a Hom-Lie deformation of the classical Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ defined by $[h, f] = -2f$, $[h, e] = 2e$, $[e, f] = h$. The Jackson $\mathfrak{sl}_2(\mathbb{K})$ is related to Jackson derivations. As linear space, it is generated by e, f, h with the brackets defined by

$$[h, f]_t = -2f - 2tf, \quad [h, e]_t = 2e, \quad [e, f]_t = h + \frac{t}{2}h.$$

The linear map α_t is defined by

$$\alpha_t(e) = \frac{2+t}{2(1+t)}e = e + \sum_{k=0}^{\infty} \frac{(-1)^k}{2} t^k e, \quad \alpha_t(h) = h, \quad \alpha_t(f) = f + \frac{t}{2}f.$$

The Hom-Jacobi identity is proved as follows. It is enough to consider it on e, f and h :

$$\begin{aligned} & [\alpha_t(e), [f, h]_t]_t + [\alpha_t(f), [h, e]_t]_t + [\alpha_t(h), [e, f]_t]_t = \\ & = (2+t)[e, f]_t + (2+t)[f, e]_t + \frac{(2+t)}{2}[h, h]_t = 0, \end{aligned}$$

where we have used that $[\cdot, \cdot]_t$ is skew-symmetric.

In this case

$$\begin{aligned} [h, f]_1 &= -2f, \quad [h, e]_1 = 0, \quad [e, f]_1 = \frac{1}{2}h, \\ \alpha_1(e) &= -\frac{1}{2}e, \quad \alpha_1(h) = 0, \quad \alpha_1(f) = \frac{1}{2}f. \end{aligned}$$

And for $k \geq 2$, one has

$$\begin{aligned} [h, f]_k &= 0, \quad [h, e]_k = 0, \quad [e, f]_k = 0, \\ \alpha_k(e) &= \frac{(-1)^k}{2}e, \quad \alpha_k(h) = 0, \quad \alpha_k(f) = 0. \end{aligned}$$

Thus Jackson $\mathfrak{sl}_2(\mathbb{K})$ algebra is a Hom-Lie algebra deformation of $\mathfrak{sl}_2(\mathbb{K})$. Indeed,

$$\begin{aligned} [h, f]_0 &= -2f, \quad [h, e]_0 = 2e, \quad [e, f]_0 = h, \\ \alpha_0(e) &= e, \quad \alpha_0(h) = h, \quad \alpha_0(f) = f. \end{aligned}$$

2.1. Deformation equation of Hom-associative algebras. In this section, we study the equation (2.1) and thus characterize the deformations of Hom-associative algebras. The equation may be written

$$(2.3) \quad \sum_{i \geq 0} \sum_{j \geq 0} \sum_{k \geq 0} (\mu_i(\alpha_k(x), \mu_j(y, z)) - \mu_i(\mu_j(x, y), \alpha_k(z))) t^{i+j+k} = 0$$

Definition 2.5. We call α -associator the map

$$(2.4) \quad Hom(V^{\times 2}, V) \times Hom(V^{\times 2}, V) \longrightarrow Hom(V^{\times 3}, V), \quad (\mu_i, \mu_j) \longmapsto \mu_i \circ_{\alpha} \mu_j,$$

defined for all $x, y, z \in V$ by

$$(2.5) \quad \mu_i \circ_{\alpha} \mu_j(x, y, z) = \mu_i(\alpha(x), \mu_j(y, z)) - \mu_i(\mu_j(x, y), \alpha(z)).$$

By using α_k -associators, the deformation equation may be written as follows

$$(2.6) \quad \sum_{i \geq 0} \sum_{j \geq 0} \sum_{k \geq 0} (\mu_i \circ_{\alpha_k} \mu_j) t^{i+j+k} = 0 \quad \text{or} \quad \sum_{s \geq 0} t^s \sum_{k=0}^s \sum_{i=0}^{s-k} \mu_i \circ_{\alpha_k} \mu_{s-k-i} = 0.$$

This equation is equivalent to the following infinite system:

$$(2.7) \quad \sum_{k=0}^s \sum_{i=0}^{s-k} \mu_i \circ_{\alpha_k} \mu_{s-k-i} = 0, \quad s = 0, 1, \dots$$

In particular,

- for $s = 0$, $\mu_0 \circ_{\alpha_0} \mu_0 = 0$, this corresponds to the Hom-associativity of \mathcal{A}_0 ;
- for $s = 1$, $\mu_1 \circ_{\alpha_0} \mu_0 + \mu_0 \circ_{\alpha_1} \mu_0 + \mu_0 \circ_{\alpha_0} \mu_1 = 0$;
- for $s = 2$, $\mu_2 \circ_{\alpha_0} \mu_0 + \mu_0 \circ_{\alpha_0} \mu_2 + \mu_1 \circ_{\alpha_0} \mu_1 + \mu_0 \circ_{\alpha_1} \mu_1 + \mu_1 \circ_{\alpha_1} \mu_0 + \mu_0 \circ_{\alpha_2} \mu_0 = 0$.

2.2. Equivalent and trivial deformations. In this section, we characterize the equivalent and trivial deformations of Hom-associative algebras.

Definition 2.6. Let $\mathcal{A}_0 = (V, \mu_0, \alpha_0)$ be a Hom-associative algebra. Given two deformations of \mathcal{A}_0 , $\mathcal{A}_t = (V, \mu_t, \alpha_t)$ and $\mathcal{A}'_t = (V, \mu'_t, \alpha'_t)$ where $\mu_t = \sum_{i \geq 0} \mu_i t^i$, $\mu'_t = \sum_{i \geq 0} \mu'_i t^i$, $\alpha_t = \sum_{i \geq 0} \alpha_i t^i$ and $\alpha'_t = \sum_{i \geq 0} \alpha'_i t^i$, with $\mu_0 = \mu'_0$ and $\alpha_0 = \alpha'_0$.

We say that they are *equivalent* if there is a formal isomorphism $\Phi_t : V \rightarrow V[[t]]$ which is a $\mathbb{K}[[t]]$ -linear map that may be written in the form $\Phi_t = \sum_{i \geq 0} \Phi_i t^i = Id + \Phi_1 t + \Phi_2 t^2 + \dots$ where $\Phi_i \in End_{\mathbb{K}}(V)$ and $\Phi_0 = Id$ such that

$$(2.8) \quad \Phi_t \circ \mu_t = \mu'_t \circ (\Phi_t \times \Phi_t) \quad \text{and} \quad \alpha'_t \circ \Phi_t = \Phi_t \circ \alpha_t.$$

A deformation \mathcal{A}_t of \mathcal{A}_0 is said to be *trivial* if and only if \mathcal{A}_t is equivalent to \mathcal{A}_0 , viewed as an algebra over $V[[t]]$.

The condition (2.8) may be written

$$(2.9) \quad \Phi_t(\mu_t(x, y)) = \mu'_t(\Phi_t(x), \Phi_t(y)), \quad \forall x, y \in V$$

and

$$(2.10) \quad \Phi_t(\alpha_t(x)) = \alpha'_t(\Phi_t(x)), \quad \forall x \in V.$$

The equation 2.9 is equivalent to

$$(2.11) \quad \sum_{i \geq 0} \Phi_i \left(\sum_{j \geq 0} \mu_j(x, y) t^j \right) t^i = \sum_{i \geq 0} \mu'_i \left(\sum_{j \geq 0} \Phi_j(x) t^j, \sum_{k \geq 0} \Phi_k(y) t^k \right) t^i$$

or

$$\sum_{i, j \geq 0} \Phi_i(\mu_j(x, y)) t^{i+j} = \sum_{i, j, k \geq 0} \mu'_i(\Phi_j(x), \Phi_k(y)) t^{i+j+k}.$$

By identification of coefficients, one obtains that the constant coefficients are identical, i.e.

$$\mu_0 = \mu'_0 \quad \text{because} \quad \Phi_0 = Id.$$

For coefficients of t one has

$$(2.12) \quad \Phi_0(\mu_1(x, y)) + \Phi_1(\mu_0(x, y)) = \mu'_1(\Phi_0(x), \Phi_0(y)) + \mu'_0(\Phi_1(x), \Phi_0(y)) + \mu'_0(\Phi_0(x), \Phi_1(y)).$$

Since $\Phi_0 = Id$, it follows that

$$(2.13) \quad \mu_1(x, y) + \Phi_1(\mu_0(x, y)) = \mu'_1(x, y) + \mu_0(\Phi_1(x), y) + \mu_0(x, \Phi_1(y)).$$

Consequently,

$$(2.14) \quad \mu'_1(x, y) = \mu_1(x, y) + \Phi_1(\mu_0(x, y)) - \mu_0(\Phi_1(x), y) - \mu_0(x, \Phi_1(y)).$$

The condition on homomorphisms (2.10) is equivalent for $x \in V$ to

$$(2.15) \quad \sum_{i,j \geq 0} \Phi_i(\alpha_j(x))t^{i+j} = \sum_{i,j \geq 0} \alpha'_i(\Phi_j(x))t^{i+j}.$$

The condition implies that $\alpha_0 = \alpha'_0 \pmod{t}$ and that

$$(2.16) \quad \Phi_1 \circ \alpha_0 + \Phi_0 \circ \alpha_1 = \alpha'_0 \circ \Phi_1 + \alpha'_1 \circ \Phi_0 \pmod{t^2}.$$

Then

$$(2.17) \quad \alpha'_1 = \alpha_1 + \Phi_1 \circ \alpha_0 - \alpha_0 \circ \Phi_1.$$

Then, The first and second order conditions of the equivalence between two deformations of a Hom-associative algebra are given by (2.14) and (2.17).

3. FIRST AND SECOND COHOMOLOGY GROUPS OF A HOM-ASSOCIATIVE ALGEBRAS

We introduce in the following certain elements of the cohomology of Hom-associative algebras which fits with the deformation theory.

3.1. First and second coboundary operators. Let $\mathcal{A} = (V, \mu, \alpha)$ be a Hom-associative algebra on a \mathbb{K} -vector space V .

The set of p -cochains on V is the set of p -linear maps

$$\mathcal{C}^p(\mathcal{A}, \mathcal{A}) = \{\varphi : V^{\times p} = \underbrace{V \times V \times \dots \times V}_{p \text{ times}} \longrightarrow V\}.$$

Definition 3.1. We set for a morphism $\tau \in Hom(V, V) = \mathcal{C}^1(\mathcal{A}, \mathcal{A})$

$$\rho_{\mathcal{A}}^1 : \mathcal{C}^1(\mathcal{A}, \mathcal{A}) \longrightarrow \mathcal{C}^1(\mathcal{A}, \mathcal{A}) \quad \tau \mapsto \rho_{\mathcal{A}}^1 \tau := \tau \circ \alpha - \alpha \circ \tau,$$

and

$$\rho_{\mathcal{A}}^2 : \mathcal{C}^2(\mathcal{A}, \mathcal{A}) \longrightarrow \mathcal{C}^3(\mathcal{A}, \mathcal{A}) \quad \tau \mapsto \rho_{\mathcal{A}}^2 \tau := \mu \circ_{\tau} \mu.$$

Now, we define 1-Hom-cochains and 2-Hom-cochains of \mathcal{A} .

Definition 3.2. A 1-Hom-cochain of \mathcal{A} is a map f , where $f \in \mathcal{C}^1(\mathcal{A}, \mathcal{A})$ satisfies

$$(3.1) \quad \rho_{\mathcal{A}}^1 f = 0.$$

We denote by $Hom\mathcal{C}^1(\mathcal{A}, \mathcal{A})$ the set of all 1-Hom-cochains of \mathcal{A} .

Definition 3.3. A 2-Hom-cochain is a pair (φ, τ) , where $\varphi \in \mathcal{C}^2(\mathcal{A}, \mathcal{A})$ and τ is a linear map such that

$$(3.2) \quad \rho_{\mathcal{A}}^2 \tau = 0.$$

We denote by $Hom\mathcal{C}^2(\mathcal{A}, \mathcal{A})$ the set of all 2-Hom-cochains of \mathcal{A} .

The 1-coboundary and 2-coboundary operators for Hom-associative algebras are defined as follows.

Definition 3.4. We call 1-coboundary operator of Hom-associative algebra \mathcal{A} the map

$$\delta_{Hom}^1 : \mathcal{C}^1(\mathcal{A}, \mathcal{A}) \longrightarrow \mathcal{C}^2(\mathcal{A}, \mathcal{A}), \quad f \mapsto \delta_{Hom}^1 f$$

defined by

$$\delta_{Hom}^1 f(x, y) = f(\mu(x, y)) - \mu(f(x), y) - \mu(x, f(y)).$$

Definition 3.5. We call a 2-coboundary operator of Hom-associative algebra \mathcal{A} the map

$$\delta_{Hom}^2 : \mathcal{C}^2(\mathcal{A}, \mathcal{A}) \longrightarrow \mathcal{C}^3(\mathcal{A}, \mathcal{A}), \quad \varphi \longmapsto \delta_{Hom}^2 \varphi$$

defined by

$$\begin{aligned} \delta_{Hom}^2 \varphi(x, y, z) &= \varphi(\alpha(x), \mu(y, z)) - \varphi(\mu(x, y), \alpha(z)) \\ &\quad + \mu(\alpha(x), \varphi(y, z)) - \mu(\varphi(x, y), \alpha(z)). \end{aligned}$$

Remark 3.6. The operator δ_{Hom}^2 can also be defined using the α -associator (2.4) by

$$\delta_{Hom}^2 \varphi = \varphi \circ_\alpha \mu + \mu \circ_\alpha \varphi.$$

The cohomology spaces relative to these coboundary operators are:

Definition 3.7. The space of 1-cohomology classes of \mathcal{A} is

$$H_{Hom}^1(\mathcal{A}, \mathcal{A}) = \{f \in Hom\mathcal{C}^1(\mathcal{A}, \mathcal{A}) : \delta_{Hom}^1 f = 0\} = \{f \in \mathcal{C}^1(\mathcal{A}, \mathcal{A}) : \delta_{Hom}^1 f = 0 \text{ and } \rho_{\mathcal{A}}^1 f = 0\}.$$

The space of 2-coboundaries of \mathcal{A} is

$$B_{Hom}^2(\mathcal{A}, \mathcal{A}) = \{(\varphi, \tau) \in Hom\mathcal{C}^2(\mathcal{A}, \mathcal{A}) : \varphi = \delta_{Hom}^1 f, f \in Hom\mathcal{C}^1(\mathcal{A}, \mathcal{A}) \text{ and } \rho_{\mathcal{A}}^2 \tau = 0\}.$$

The space of 2-cocycles of \mathcal{A} is

$$Z_{Hom}^2(\mathcal{A}, \mathcal{A}) = \{(\varphi, \tau) \in Hom\mathcal{C}^2(\mathcal{A}, \mathcal{A}) : \delta_{Hom}^2 \varphi = 0 \text{ and } \rho_{\mathcal{A}}^2 \tau = 0\}.$$

Proposition 3.8. $\delta_{Hom}^2(\delta_{Hom}^1) = 0$.

Proof. Let

$$\delta_{Hom}^1 f(x, y) = f(\mu(x, y)) - \mu(f(x), y) - \mu(x, f(y)).$$

Then

$$\begin{aligned} \delta_{Hom}^2(\delta_{Hom}^1 f)(x, y, z) &= \delta_{Hom}^1 f(\alpha(x), \mu(y, z)) - \delta_{Hom}^1 f(\mu(x, y), \alpha(z)) \\ &\quad + \mu(\alpha(x), \delta_{Hom}^1 f(y, z)) - \mu(\delta_{Hom}^1 f(x, y), \alpha(z)) \\ &= f(\mu(\alpha(x), \mu(y, z))) - \mu(f(\alpha(x)), \mu(y, z)) - \mu(\alpha(x), f(\mu(y, z))) \\ &\quad - f(\mu(\mu(x, y), \alpha(z))) + \mu(f(\mu(x, y)), \alpha(z)) + \mu(\mu(x, y), f(\alpha(z))) \\ &\quad + \mu(\alpha(x), f(\mu(y, z))) - \mu(\alpha(x), \mu(f(y), z)) - \mu(\alpha(x), \mu(y, f(z))) \\ &\quad - \mu(f(\mu(x, y)), \alpha(z)) + \mu(\mu(f(x), y), \alpha(z)) + \mu(\mu(x, f(y)), \alpha(z)) \\ &= 0, \end{aligned}$$

because α and f commute and the multiplication μ is Hom-associative. □

Remark 3.9. One has $B_{Hom}^2(\mathcal{A}, \mathcal{A}) \subset Z_{Hom}^2(\mathcal{A}, \mathcal{A})$, because $\delta_{Hom}^2 \circ \delta_{Hom}^1 = 0$. Note also that $H_{Hom}^1(\mathcal{A}, \mathcal{A})$ corresponds to the derivations space of a Hom-associative algebra \mathcal{A} .

Definition 3.10. We call the 2^{th} cohomology group of the Hom-associative algebra \mathcal{A} , the quotient

$$H_{Hom}^2(\mathcal{A}, \mathcal{A}) = \frac{Z_{Hom}^2(\mathcal{A}, \mathcal{A})}{B_{Hom}^2(\mathcal{A}, \mathcal{A})}.$$

Remark 3.11. The cohomology class of an element (φ, τ) is given by the set of elements of the form (ψ, τ) such that $\psi = \delta_{Hom}^2 f$ where f is a 1-Hom-cochain, that is $f \in \mathcal{C}^1(\mathcal{A}, \mathcal{A})$ and $\rho_{\mathcal{A}}^1 f = 0$.

4. COHOMOLOGICAL APPROACH OF HOM-ASSOCIATIVE ALGEBRA DEFORMATIONS

Let $\mathcal{A}_t = (V, \mu_t, \alpha_t)$ be a deformation of a Hom-associative algebra $\mathcal{A}_0 = (V, \mu_0, \alpha_0)$ where $\mu_t(x, y) = \sum_{i \geq 0} \mu_i(x, y)t^i$ and $\alpha_t(x) = \sum_{i \geq 0} \alpha_i(x)t^i$. We characterize under the assumption $\rho_{\mathcal{A}}^2 \alpha_i = \mu_0 \circ_{\alpha_i} \mu_0 = 0$ for $i \geq 1$ the deformations of \mathcal{A}_0 in terms of cohomology.

By using the definition of 2-coboundaries and by gathering some terms, the deformation equation (2.7) may be written

$$\delta_{Hom}^2 \mu_1 = 0$$

and

$$(4.1) \quad \delta_{Hom}^2 \mu_s = - \sum_{k=1}^{s-1} \sum_{p=0}^k \mu_{s-k} \circ_{\alpha_p} \mu_{k-p} - \sum_{k=1}^{s-k} \mu_0 \circ_{\alpha_k} \mu_{s-k}, \quad s = 2, 3, \dots$$

Consequently, the following Lemma holds.

Lemma 4.1. *The pair (μ_1, α_1) of the deformation \mathcal{A}_t is a 2-Hom-cocycle of the cohomology of the Hom-associative algebra \mathcal{A}_0 .*

Definition 4.2. Let $\mathcal{A}_0 = (V, \mu_0, \alpha_0)$ be a Hom-associative algebra and (μ_1, α_1) be an element of $Z_{Hom}^2(\mathcal{A}_0, \mathcal{A}_0)$. The 2-Hom-cocycle (μ_1, α_1) is said *integrable* if there exists a pair (μ_t, α_t) such that $\mu_t = \sum_{i \geq 0} \mu_i t^i$ and $\alpha_t = \sum_{i \geq 0} \alpha_i t^i$ defining a deformation $\mathcal{A}_t = (V, \mu_t, \alpha_t)$ of \mathcal{A}_0 .

Proposition 4.3. *Let $\mathcal{A}_t = (V, \mu_t, \alpha_t)$ be a deformation of a Hom-associative algebra \mathcal{A}_0 . Let $\mu_t = \sum_{i \geq 0} \mu_i t^i$, $\alpha_t = \sum_{i \geq 0} \alpha_i t^i$ and (μ_1, α_1) be an element of $Z_{Hom}^2(\mathcal{A}_0, \mathcal{A}_0)$. The integrability of (μ_1, α_1) depends only on its cohomology class.*

Proof. We saw in Section 2 that if two deformations $\mathcal{A}_t = (V, \mu_t, \alpha_t)$ and $\mathcal{A}'_t = (V, \mu'_t, \alpha'_t)$ are equivalent then

$$\mu'_1(x, y) = \mu_1(x, y) + \Phi_1(\mu_0(x, y)) - \mu_0(\Phi_1(x), y) - \mu_0(x, \Phi_1(y))$$

and

$$\alpha'_1 = \alpha_1 + \Phi_1 \circ \alpha_0 - \alpha_0 \circ \Phi_1.$$

With $\alpha'_1 = \alpha_1$, these conditions means that

$$\mu'_1 = \mu_1 + \delta_{Hom}^1 \Phi_1 \quad \text{and} \quad \rho_{\mathcal{A}}^1(\Phi) = 0.$$

Therefore the two elements are cohomologous.

Thus,

$$\begin{aligned} \delta_{Hom}^2 \mu_1 = 0 &\implies \delta_{Hom}^2 \mu'_1 = \delta_{Hom}^2 (\mu_1 + \delta_{Hom}^1 \Phi_1) = \delta_{Hom}^2 \mu_1 + \delta_{Hom}^2 (\delta_{Hom}^1 \Phi_1) = 0; \\ \mu_1 = \delta_{Hom}^1 g &\implies \mu'_1 = \delta_{Hom}^1 g - \delta_{Hom}^1 \Phi_1 = \delta_{Hom}^1 (g - \Phi_1); \end{aligned}$$

which ends the proof. \square

Proposition 4.4. *Let $\mathcal{A}_0 = (V, \mu_0, \alpha_0)$ be a Hom-associative algebra. There is, over $\mathbb{K}[[t]]/t^2$, a one-to-one correspondence between the elements of $H_{Hom}^2(\mathcal{A}_0, \mathcal{A}_0)$ and the infinitesimal deformation of \mathcal{A}_0 defined by*

$$(4.2) \quad \mu_t(x, y) = \mu_0(x, y) + \mu_1(x, y)t, \quad \text{and} \quad \alpha_t(x) = \alpha_0(x) + \alpha_1(x)t \quad \forall x, y \in V$$

with $\rho_{\mathcal{A}}^2(\alpha_1) = \mu_0 \circ_{\alpha_1} \mu_0 = 0$

Proof. The deformation equation is equivalent to

$$\delta_{Hom}^2 \mu_1 + \rho_{\mathcal{A}}^2(\alpha_1) = 0.$$

Since $\rho_{\mathcal{A}}^2(\alpha_1) = 0$, then the previous equation is equivalent to $(\mu_1, \alpha_1) \in Z_{Hom}^2(\mathcal{A}_0, \mathcal{A}_0)$. \square

4.1. Poisson algebra. We consider now a commutative Hom-associative algebra and show that first order deformation induces a Hom-Poisson structure (Definition 1.4). More generally, the following lemma shows that any skewsymmetric 2-Hom-cocycle of a commutative algebra satisfies the compatibility condition 1.4 with the multiplication of the Hom-associative algebra.

Lemma 4.5. *Let $\mathcal{A} = (V, \mu, \alpha)$ be a commutative Hom-associative algebra and φ be a skewsymmetric 2-cochain such that $\delta_{Hom}^2 \varphi = 0$. Then for $x, y, z \in V$*

$$(4.3) \quad \varphi(\alpha(x), \mu(y, z)) = \mu(\alpha(y), \varphi(x, z)) + \mu(\alpha(z), \varphi(x, y)).$$

Proof. The condition $\delta_{Hom}^2 \varphi = 0$ is equivalent to

$$\varphi(\alpha(x), \mu(y, z)) - \varphi(\mu(x, y), \alpha(z)) + \mu(\alpha(x), \varphi(y, z)) - \mu(\varphi(x, y), \alpha(z)) = 0$$

Then one has

$$(4.4) \quad \varphi(\alpha(x), \mu(y, z)) = \varphi(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \varphi(y, z)) + \mu(\varphi(x, y), \alpha(z))$$

$$(4.5) \quad \varphi(\alpha(x), \mu(z, y)) = \varphi(\mu(x, z), \alpha(y)) - \mu(\alpha(x), \varphi(z, y)) + \mu(\varphi(x, z), \alpha(y))$$

$$(4.6) \quad \varphi(\mu(y, x), \alpha(z)) - \varphi(\alpha(y), \mu(x, z)) = \mu(\alpha(y), \varphi(x, z)) - \mu(\varphi(y, x), \alpha(z)).$$

By adding the equations 4.4, 4.5 and 4.6 and considering the fact that φ is skewsymmetric and μ is commutative one has

$$2 \varphi(\alpha(x), \mu(y, z)) = 2 \mu(\alpha(y), \varphi(x, z)) + 2 \mu(\alpha(z), \varphi(x, y)).$$

□

Let $\mathcal{A}_t = (V, \mu_t, \alpha_t)$ be a deformation of the commutative Hom-associative algebra $\mathcal{A}_0 = (V, \mu_0, \alpha_0)$. Assume that

$$\mu_t(x, y) = \mu_0(x, y) + \mu_1(x, y)t + \mu_2(x, y)t^2 + \dots, \quad \forall x, y \in V$$

and

$$\alpha_t(x) = \alpha_0(x) + \alpha_1(x)t + \alpha_2(x)t^2 + \dots, \quad \forall x \in V.$$

Then

$$\frac{\mu_t(x, y) - \mu_t(y, x)}{t} = \mu_1(x, y) - \mu_1(y, x) + t \sum_{i \geq 2} (\mu_i(x, y) - \mu_i(y, x))t^{i-1}.$$

Hence, if t goes to zero then $\frac{\mu_t(x, y) - \mu_t(y, x)}{t}$ goes to $\{x, y\} := \mu_1(x, y) - \mu_1(y, x)$. The previous bracket will define a structure of Poisson algebra over the commutative algebra \mathcal{A}_0 .

Lemma 4.6. *Let $\mathcal{A}_0 = (V, \mu_0, \alpha_0)$ be a commutative Hom-associative algebra and $\mathcal{A}_t = (V, \mu_t, \alpha_t)$ be a deformation of \mathcal{A}_0 . Then*

$$\odot_{x, y, z} \delta_{Hom}^2 \mu_2(x, y, z) = \odot_{x, y, z} \mu_2 \circ_{\alpha_0} \mu_0(x, y, z).$$

Proof.

$$\begin{aligned} \odot_{x, y, z} \delta_{Hom}^2 \mu_2(x, y, z) &= \mu_2(\alpha_0(x), \mu_0(y, z)) - \mu_2(\mu_0(x, y), \alpha_0(z)) \\ &\quad + \mu_0(\alpha_0(x), \mu_2(y, z)) - \mu_0(\mu_2(x, y), \alpha_0(z)) \\ &\quad + \mu_2(\alpha_0(y), \mu_0(z, x)) - \mu_2(\mu_0(y, z), \alpha_0(x)) \\ &\quad + \mu_0(\alpha_0(y), \mu_2(z, x)) - \mu_0(\mu_2(y, z), \alpha_0(x)) \\ &\quad + \mu_2(\alpha_0(z), \mu_0(x, y)) - \mu_2(\mu_0(z, x), \alpha_0(y)) \\ &\quad + \mu_0(\alpha_0(z), \mu_2(x, y)) - \mu_0(\mu_2(z, x), \alpha_0(y)) \\ &= \odot_{x, y, z} \mu_2 \circ_{\alpha_0} \mu_0(x, y, z). \end{aligned}$$

□

Lemma 4.7. *Let $\mathcal{A}_0 = (V, \mu_0, \alpha_0)$ be a commutative Hom-associative algebra and $\mathcal{A}_t = (V, \mu_t, \alpha_0)$ be a deformation of \mathcal{A}_0 . Then*

$$\circlearrowleft_{x,y,z} \delta_{Hom}^2 \mu_2(x, y, z) - \circlearrowleft_{x,z,y} \delta_{Hom}^2 \mu_2(x, z, y) = 0.$$

Proof.

$$\begin{aligned} \circlearrowleft_{x,y,z} \delta_{Hom}^2 \mu_2(x, y, z) - \circlearrowleft_{x,z,y} \delta_{Hom}^2 \mu_2(x, z, y) &= \circlearrowleft_{x,y,z} \mu_2 \circ_{\alpha_0} \mu_0(x, y, z) - \circlearrowleft_{x,z,y} \mu_2 \circ_{\alpha_0} \mu_0(x, z, y) \\ &= \mu_2(\alpha_0(x), \mu_0(y, z)) - \mu_2(\mu_0(x, y), \alpha_0(z)) \\ &\quad + \mu_0(\alpha_0(y), \mu_2(z, x)) - \mu_0(\mu_2(y, z), \alpha_0(x)) \\ &\quad + \mu_2(\alpha_0(z), \mu_0(x, y)) - \mu_2(\mu_0(z, x), \alpha_0(y)) \\ &\quad - \mu_2(\alpha_0(x), \mu_0(z, y)) + \mu_2(\mu_0(x, z), \alpha_0(y)) \\ &\quad - \mu_2(\alpha_0(z), \mu_0(y, x)) + \mu_2(\mu_0(z, y), \alpha_0(x)) \\ &\quad - \mu_2(\alpha_0(y), \mu_0(x, z)) - \mu_2(\mu_0(y, x), \alpha_0(z)) \\ &= 0. \end{aligned}$$

□

Lemma 4.8. *Let $\mathcal{A}_0 = (V, \mu_0, \alpha_0)$ be a commutative Hom-associative algebra and α any linear map of $Hom(V, V)$. Then*

$$\circlearrowleft_{x,y,z} \mu_0 \circ_{\alpha} \mu_0(x, y, z) = 0.$$

Proof.

$$\begin{aligned} \circlearrowleft_{x,y,z} \mu_0 \circ_{\alpha} \mu_0(x, y, z) &= \mu_0(\alpha(x), \mu_0(y, z)) - \mu_0(\mu_0(x, y), \alpha(z)) \\ &\quad + \mu_0(\alpha(y), \mu_0(z, x)) - \mu_0(\mu_0(y, z), \alpha(x)) \\ &\quad + \mu_0(\alpha(z), \mu_0(x, y)) - \mu_0(\mu_0(z, x), \alpha(y)) \\ &= 0. \end{aligned}$$

□

Theorem 4.9. *Let $\mathcal{A}_0 = (V, \mu_0, \alpha_0)$ be a commutative Hom-associative algebra and $\mathcal{A}_t = (V, \mu_t, \alpha_t)$ be a deformation of \mathcal{A}_0 . Consider the bracket defined for $x, y \in V$ by $\{x, y\} = \mu_1(x, y) - \mu_1(y, x)$ where μ_1 is the first order element of the deformation μ_t .*

Then $(V, \mu_0, \{\cdot, \cdot\}, \alpha_0)$ is a Hom-Poisson algebra.

Proof. The bracket is skewsymmetric by definition. The compatibility condition follows from Lemma 4.6. Let us prove that the Hom-Jacobi condition is satisfied by the bracket. One has

$$\begin{aligned} \circlearrowleft_{x,y,z} \{\alpha_0(x), \{y, z\}\} &= \circlearrowleft_{x,y,z} (\mu_1(\alpha_0(x), \mu_1(y, z)) - \mu_1(\alpha_0(x), \mu_1(z, y)) + \\ &\quad - \mu_1(\mu_1(y, z), \alpha(x)) + \mu_1(\mu_1(z, y), \alpha(x))) \\ &= \mu_1(\alpha_0(x), \mu_1(y, z)) - \mu_1(\alpha_0(x), \mu_1(z, y)) + \\ &\quad - \mu_1(\mu_1(y, z), \alpha(x)) + \mu_1(\mu_1(z, y), \alpha(x)) \\ &\quad + \mu_1(\alpha_0(y), \mu_1(z, x)) - \mu_1(\alpha_0(y), \mu_1(x, z)) + \\ &\quad - \mu_1(\mu_1(z, x), \alpha(y)) + \mu_1(\mu_1(x, z), \alpha(y)) \\ &\quad + \mu_1(\alpha_0(z), \mu_1(x, y)) - \mu_1(\alpha_0(z), \mu_1(y, x)) + \\ &\quad - \mu_1(\mu_1(x, y), \alpha(z)) + \mu_1(\mu_1(y, x), \alpha(z)) \\ &= \circlearrowleft_{x,y,z} \mu_1 \circ_{\alpha_0} \mu_1(x, y, z) - \circlearrowleft_{x,z,y} \mu_1 \circ_{\alpha_0} \mu_1(x, z, y). \end{aligned}$$

The deformation equation (2.7) implies for $s = 2$ that

$$\mu_1 \circ_{\alpha_0} \mu_1 = -\mu_2 \circ_{\alpha_0} \mu_0 - \mu_0 \circ_{\alpha_0} \mu_2 - \mu_1 \circ_{\alpha_1} \mu_0 - \mu_0 \circ_{\alpha_1} \mu_1 - \mu_0 \circ_{\alpha_2} \mu_0$$

which is equivalent to

$$\mu_1 \circ_{\alpha_0} \mu_1 = -\delta_{Hom}^2 \mu_2 - \mu_0 \circ_{\alpha_1} \mu_1 - \mu_1 \circ_{\alpha_1} \mu_0 - \mu_0 \circ_{\alpha_2} \mu_0$$

Then using the previous Lemmas

$$\begin{aligned} \circlearrowleft_{x,y,z} \{\alpha(x), \{y, z\}\} &= \circlearrowleft_{x,y,z} (-\delta_{Hom}^2 \mu_2(x, y, z) - \mu_0 \circ_{\alpha_1} \mu_1(x, y, z) - \mu_1 \circ_{\alpha_1} \mu_0(x, y, z) - \mu_0 \circ_{\alpha_2} \mu_0(x, y, z)) \\ &\quad + \circlearrowleft_{x,z,y} (\delta_{Hom}^2 \mu_2(x, z, y) + \mu_0 \circ_{\alpha_1} \mu_1(x, z, y) + \mu_1 \circ_{\alpha_1} \mu_0(x, z, y) + \mu_0 \circ_{\alpha_2} \mu_0(x, z, y)) \\ &= 0 \end{aligned}$$

□

5. ON COHOMOLGY OF HOM-LIE ALGEBRAS

Now we introduce elements of cohomology of Hom-Lie algebras in connection to their infinitesimal deformations.

5.1. First and second coboundary operators. Let $\mathcal{G} = (V, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra on a \mathbb{K} -vector space V .

The set of p -cochains on V is the set of p -linear alternating maps

$$\mathcal{C}^p(\mathcal{G}, \mathcal{G}) = \{\varphi : V^{\wedge p} = \underbrace{V \wedge V \wedge \dots \wedge V}_{p \text{ times}} \longrightarrow V\}$$

In the following, we define 1-Hom-cochains and 2-Hom-cochains.

Definition 5.1. A 1-Hom-cochain is a map f , where $f \in \mathcal{C}^1(\mathcal{G}, \mathcal{G})$ satisfying

$$(5.1) \quad f \circ \alpha = \alpha \circ f$$

We denote by $Hom\mathcal{C}^1(\mathcal{G}, \mathcal{G})$ the set of all 1-Hom-cochain of \mathcal{G} .

Definition 5.2. A 2-Hom-cochain is a pair (φ, τ) , where $\varphi \in \mathcal{C}^2(\mathcal{G}, \mathcal{G})$ is a 2-linear alternating map and τ is a linear map satisfying

$$(5.2) \quad \circlearrowleft_{x,y,z} [\tau(x), [y, z]] = 0$$

We denote by $Hom\mathcal{C}^2(\mathcal{G}, \mathcal{G})$ the set of all 2-Hom-cochains of \mathcal{G} .

The 1-coboundary and 2-coboundary operators for Hom-Lie algebras are defined as follows

Definition 5.3. We call 1-coboundary operator of Hom-Lie algebra \mathcal{G} the map

$$\delta_{HL}^1 : \mathcal{C}^1(\mathcal{G}, \mathcal{G}) \longrightarrow \mathcal{C}^2(\mathcal{G}, \mathcal{G}), \quad f \longmapsto \delta_{HL}^1 f$$

defined by

$$\delta_{HL}^1 f(x, y) = f([x, y]) - [f(x), y] - [x, f(y)]$$

Definition 5.4. We call a 2-coboundary operator of Hom-Lie algebra \mathcal{G} the map

$$\delta_{HL}^2 : \mathcal{C}^2(\mathcal{G}, \mathcal{G}) \longrightarrow \mathcal{C}^3(\mathcal{G}, \mathcal{G}), \quad \varphi \longmapsto \delta_{HL}^2 \varphi$$

defined by

$$\delta_{HL}^2 \varphi(x, y, z) = \circlearrowleft_{x,y,z} \varphi(\alpha(x), [y, z]) + [\alpha(x), \varphi(y, z)]$$

The cohomology spaces relative to these coboundary operators are

Definition 5.5. The space of 1-cocycles of \mathcal{A} is

$$H_{HL}^1(\mathcal{G}, \mathcal{G}) = \{f \in Hom\mathcal{C}^1(\mathcal{G}, \mathcal{G}) : \delta_{HL}^1 f = 0\} = \{f \in \mathcal{C}^1(\mathcal{G}, \mathcal{G}) : \delta_{HL}^1 f = 0 \text{ and } f \circ \alpha = \alpha \circ f\}$$

The space of 2-coboundaries of \mathcal{A} is

$$B_{HL}^2(\mathcal{G}, \mathcal{G}) = \{(\varphi, \tau) \in Hom\mathcal{C}^2(\mathcal{G}, \mathcal{G}) : \varphi = \delta_{HL}^1 f, f \in Hom\mathcal{C}^1(\mathcal{G}, \mathcal{G}) \text{ and } \circlearrowleft_{x,y,z} [\tau(x), [y, z]] = 0 \forall x, y, z \in V\}$$

The space of 2-cocycles of \mathcal{G} is

$$Z_{HL}^2(\mathcal{G}, \mathcal{G}) = \{(\varphi, \tau) \in Hom\mathcal{C}^2(\mathcal{G}, \mathcal{G}) : \delta_{HL}^2 \varphi = 0 \text{ and } \circlearrowleft_{x,y,z} [\tau(x), [y, z]] = 0 \forall x, y, z \in V\}$$

Proposition 5.6. $\delta_{HL}^2(\delta_{HL}^1) = 0$

Proof. Let

$$\delta_{HL}^1 f(x, y) = f([x, y]) - [f(x), y] - [x, f(y)]$$

One has

$$\begin{aligned} \delta_{HL}^2(\delta_{HL}^1 f)(x, y, z) &= f([\alpha(x), [y, z]]) - [f(\alpha(x)), [y, z]] - [\alpha(x), f([y, z])] \\ &\quad + [\alpha(x), f([y, z])] - [\alpha(x), [f(y), z]] - [\alpha(x), [y, f(z)]] \\ &\quad f([\alpha(y), [z, x]]) - [f(\alpha(y)), [z, x]] - [\alpha(y), f([z, x])] \\ &\quad + [\alpha(y), f([z, x])] - [\alpha(y), [f(z), x]] - [\alpha(y), [z, f(x)]] \\ &\quad f([\alpha(z), [x, y]]) - [f(\alpha(z)), [x, y]] - [\alpha(z), f([x, y])] \\ &\quad + [\alpha(z), f([x, y])] - [\alpha(z), [f(x), y]] - [\alpha(z), [x, f(y)]]. \end{aligned}$$

Since α and f commute and the bracket satisfies the Hom-Jacobi identity, then

$$\begin{aligned} \delta_{HL}^2(\delta_{HL}^1 f)(x, y, z) &= f(\circlearrowleft_{x,y,z} [\alpha(x), [y, z]]) - \circlearrowleft_{f(x),y,z} [\alpha(f(x)), [y, z]] \\ &\quad - \circlearrowleft_{x,f(y),z} [\alpha(x), [f(y), z]] - \circlearrowleft_{x,y,f(z)} [\alpha(x), [y, f(z)]] = 0. \end{aligned}$$

□

Remark 5.7. One has $B_{HL}^2(\mathcal{G}, \mathcal{G}) \subset Z_{HL}^2(\mathcal{G}, \mathcal{G})$, because $\delta_{HL}^2 \circ \delta_{HL}^1 = 0$. Note also that $Z_{HL}^1(\mathcal{G}, \mathcal{G})$ gives the space of derivations of a Hom-Lie algebra \mathcal{G} , denoted $Der_{HL}(\mathcal{G})$.

Definition 5.8. We call the 2^{th} cohomology group of the Hom-Lie algebra \mathcal{A} the quotient

$$H_{HL}^2(\mathcal{G}, \mathcal{G}) = \frac{Z_{HL}^2(\mathcal{G}, \mathcal{G})}{B_{HL}^2(\mathcal{G}, \mathcal{G})}.$$

5.2. Deformations of Hom-Lie algebras in terms of cohomology. The results in this section are similar to those obtained for Hom-associative algebra in Section 4.

Let $\mathcal{G} = (V, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra on a \mathbb{K} -vector space V and $\mathcal{G}_t = (V, [\cdot, \cdot]_t, \alpha_t)$ where $[x, y]_t = \sum_{i \geq 0} [x, y]_i t^i$ and $\alpha_t(x) = \sum_{i \geq 0} \alpha_i(x) t^i$, $x, y \in V$, such that $[\cdot, \cdot]_i$ are bilinear alternating maps with $[\cdot, \cdot]_0$ being the bracket of \mathcal{G} and $\alpha_0 = \alpha$. We assume in this Section that the deformation satisfies $\circlearrowleft_{x,y,z} [\alpha_1(x), [y, z]_0]_0 = 0$.

Proposition 5.9. *The first order term $([\cdot, \cdot]_1, \alpha_1)$ of the deformation $\mathcal{G}_t = (V, [\cdot, \cdot]_t, \alpha_t)$ of $\mathcal{G} = (V, [\cdot, \cdot], \alpha)$ satisfying $\circlearrowleft_{x,y,z} [\alpha_1(x), [y, z]_0]_0 = 0$ is a 2-Hom-cocycle of the Hom-Lie algebra \mathcal{G} .*

Proof. The deformation equation 2.2 implies

$$\sum_{i+j+k=s} \circlearrowleft_{x,y,z} [\alpha_j(x), [y, z]_k]_i = 0, \text{ for } s > 0$$

For $s = 1$, one has

$$\circlearrowleft_{x,y,z} [\alpha_0(x), [y, z]_0]_1 + [\alpha_1(x), [y, z]_0]_0 + [\alpha_0(x), [y, z]_1]_0 = 0.$$

Since $\circlearrowleft_{x,y,z} [\alpha_1(x), [y, z]_0]_0 = 0$ then for $\psi = [\cdot, \cdot]_1$, $\delta_{HL}^2 \psi = 0$. □

6. EXAMPLES OF HOM-LIE DEFORMATIONS OF $\mathfrak{sl}_2(\mathbb{K})$

In this section, we construct all the twistings so that the $\mathfrak{sl}_2(\mathbb{K})$ brackets determine a Hom-Lie algebra, provide families of Hom-Lie algebras deforming the $\mathfrak{sl}_2(\mathbb{K})$ Lie algebra and show that q -deformed Witt algebras may be viewed as a Hom-deformation.

6.1. Infinitesimal Hom-Lie deformations of $\mathfrak{sl}_2(\mathbb{K})$. In this section, we deform the $\mathfrak{sl}_2(\mathbb{K})$ Lie algebra as a Hom-Lie algebra. The following proposition gives all the twistings so that the brackets $[X_1, X_2] = 2X_2$, $[X_1, X_3] = -2X_3$, $[X_2, X_3] = X_1$ determine a three dimensional Hom-Lie algebra, generalizing $\mathfrak{sl}_2(\mathbb{K})$.

Proposition 6.1. *Let V be a three dimensional \mathbb{K} -linear space and let (x_1, x_2, x_3) be its basis. Any Hom-Lie algebra with the following brackets*

$$[x_1, x_2] = 2x_2, [x_1, x_3] = -2x_3, [x_2, x_3] = x_1$$

is given by linear α defined, with respect to the previous basis, by a matrix of the form
$$\begin{pmatrix} a & d & c \\ 2c & b & f \\ 2d & e & b \end{pmatrix}$$

where a, c, d, e, f are arbitrary parameters in \mathbb{K}

Remark 6.2. The Hom-Lie algebras given by the previous theorem are deformations of $\mathfrak{sl}_2(\mathbb{K})$ viewed as a Hom-Lie algebra where α_0 is the identity matrix.

In the following we provide infinitesimal Hom-Lie deformations of $\mathfrak{sl}_2(\mathbb{K})$. We construct 3-dimensional Hom-Lie algebras defined by the bracket $[\cdot, \cdot]_t = [\cdot, \cdot]_0 + t[\cdot, \cdot]_1$ and homomorphism $\alpha_t = \alpha_0 + t\alpha_1$, where α_0 is the identity map, satisfying Hom-Jacobi condition and such that $([\cdot, \cdot]_1, \alpha_1)$ is a 2-Hom-cocycle for the cohomology Hom-Lie algebras defined above.

The following pairs $([\cdot, \cdot]_1, \alpha_1)$ define a 2-Hom-cocycle and thus corresponding infinitesimal deformations of $\mathfrak{sl}_2(\mathbb{K})$.

$$(1) \quad \begin{aligned} [x_1, x_2]_1 &= -a_1x_2 + x_3 \\ [x_1, x_3]_1 &= a_2x_2 + a_1x_3, \\ [x_2, x_3]_1 &= a_3x_1, \end{aligned} \quad \alpha_1 = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & -b_3a_2 \\ 0 & b_3 & b_2 \end{pmatrix}$$

where a_1, a_2, a_3, b_1, b_2 are parameters in \mathbb{K} .

$$(2) \quad \begin{aligned} [x_1, x_2]_1 &= -2a_1x_2 \\ [x_1, x_3]_1 &= a_2x_2 + 2a_1x_3, \\ [x_2, x_3]_1 &= -a_1x_1, \end{aligned} \quad \alpha_1 = \begin{pmatrix} b_1 & 0 & b_3 \\ 2b_3 & b_2 & b_4 \\ 0 & 0 & b_2 \end{pmatrix}$$

where $a_1, a_2, b_1, b_2, b_3, b_4$ are parameters in \mathbb{K} .

$$(3) \quad \begin{aligned} [x_1, x_2]_1 &= -a_1x_2 + a_2x_3 \\ [x_1, x_3]_1 &= a_3x_1 + a_4x_2 + a_1x_3, \\ [x_2, x_3]_1 &= a_5x_1 - a_3x_2, \end{aligned} \quad \alpha_1 = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}$$

where $a_1, a_2, a_3, a_4, a_5, b$ are parameters in \mathbb{K} with $a_3 \neq 0$.

Remark 6.3. The above three examples are actually Lie algebras for all values of parameters a_j and b_j since b_j 's can be chosen arbitrary and in particular so that $\alpha_1 = id$ and hence $\alpha_0 + t\alpha_1 = (1+t)id$ corresponding to the case of Lie algebra. We have also computed with a computer algebra system "Mathematica" many other examples of infinitesimal formal Hom-Lie deformations of $\mathfrak{sl}_2(\mathbb{K})$ with additional restriction that $(V, [\cdot, \cdot]_0, \alpha_1)$ is a Hom-Lie algebra. Remarkably, it turns out that in all these examples the Hom-Lie algebras are actually Lie algebras. We conjecture that this is always the case for such Hom-Lie infinitesimal formal deformations of $\mathfrak{sl}_2(\mathbb{K})$.

Now, we will give examples of Hom-Lie infinitesimal formal deformations of $\mathfrak{sl}_2(\mathbb{K})$ which are not Lie algebras. We consider the 3-dimensional Hom-Lie algebras with the bracket $[\cdot, \cdot]_t$ and linear map α_t defined as follows

$$\begin{aligned} [x_1, x_2]_t &= a_1tx_1 + (2 - a_2t)x_2 \\ [x_1, x_3]_t &= a_3tx_1 + a_4tx_2 + (-2 + a_2t)x_3, \\ [x_2, x_3]_t &= (1 - \frac{a_2}{2}t)x_1, \end{aligned} \quad \alpha_t = \begin{pmatrix} 1 + b_1t & \frac{a_1}{2}t & \frac{b_2 - a_3}{2}t \\ b_2t & 1 - \frac{a_2}{2}t & -\frac{a_4}{2}t \\ 0 & 0 & 1 - \frac{a_2}{2}t \end{pmatrix}$$

where $a_1, a_2, a_3, a_4, b_1, b_2$ are parameters in \mathbb{K} . This Hom-Lie algebra becomes a Lie algebra for all t if and only if $a_1 = 0$ and $a_3 = 0$, as follows from

$$[x_1, [x_2, x_3]] + [x_3, [x_1, x_2]] + [x_2, [x_3, x_1]] = (2a_3t - (a_2a_3 + a_1a_4)t^2)x_2 + (2a_1t - a_1a_2t^2)x_3.$$

6.2. q -deformed Witt algebras. Let \mathcal{A} be the unique factorization domain $\mathbb{K}[z, z^{-1}]$, the Laurent polynomials in t over the field \mathbb{K} . Then the space $\mathcal{D}_\sigma(\mathcal{A})$ can be generated by a single element D as a left \mathcal{A} -module, that is, $\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{A} \cdot D$ (Theorem 1 in [12]). When $\sigma(z) = qz$ with $q \neq 0$ and $q \neq 1$, one can take D as z times the Jackson q -derivative

$$D = \frac{id - \sigma}{1 - q} \quad : \quad f(t) \mapsto \frac{f(z) - f(qz)}{1 - q}.$$

The \mathbb{K} -linear space $\mathcal{D}_\sigma(\mathcal{A}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{K} \cdot x_n$, with $x_n = -z^n D$ can be equipped with the skew-symmetric bracket $[\cdot, \cdot]_\sigma$ defined on generators as

$$(6.1) \quad [x_n, x_m] = q^n x_n x_m - q^m x_m x_n = [x_n, x_m] = (\{n\}_q - \{m\}_q) x_{n+m},$$

where $\{n\}_q = (q^n - 1)/(q - 1)$ for $q \neq 1$ and $\{n\}_1 = n$. This bracket is skew-symmetric and satisfies the σ -deformed Jacobi-identity

$$(6.2) \quad (q^n + 1)[x_n, [x_l, x_m]] + (q^l + 1)[x_l, [x_m, x_n]_\sigma] + (q^m + 1)[x_m, [x_n, x_l]] = 0.$$

We have a Hom-Lie algebra $(V, [\cdot, \cdot], \alpha) = (\bigoplus_{i \in \mathbb{Z}} \mathbb{K} x_i, [\cdot, \cdot], \alpha)$ with the bilinear bracket defined on generators as $[x_n, x_m] = (\{n\}_q - \{m\}_q) x_{n+m}$ and the linear twisting map $\alpha : V \rightarrow V$ acting on generators as $\alpha(x_n) = (q^n + 1)x_n$. Obviously this Hom-Lie algebra can be viewed as a q -deformed Witt algebra in the sense that for $q = 1$ indeed one recovers the bracket and the commutation relations for generators of the Witt algebra. The definition of its generators using first order differential operators is recovered if one assumes that $D = t \frac{d}{dt}$ for $q = 1$ as one would expect from passing to a limit in the definition of the operator D .

It can be also shown that there is a central extension Vir_q of this deformation in the category of hom-Lie algebras [12], therefore being a natural q -deformation of the Virasoro algebra. The algebra Vir_q is spanned by elements $\{x_n \mid n \in \mathbb{Z}\} \cup \{\mathbf{c}\}$ where \mathbf{c} is central with respect to Hom-Lie bracket, i.e., $[Vir_q, \mathbf{c}] = [\mathbf{c}, Vir_q] = 0$. The bracket of x_n, x_m is computed according to

$$[x_n, x_m] = (\{n\}_q - \{m\}_q) x_{n+m} + \delta_{n+m,0} \frac{q^{-n}}{6(1+q^n)} \{n-1\}_q \{n\}_q \{n+1\}_q \mathbf{c}.$$

Note that when $q = 1$ we retain the classical Virasoro algebra

$$[x_n, x_m] = (n - m) x_{n+m} + \delta_{n+m,0} \frac{1}{12} \{n-1\} \{n\} \{n+1\} \mathbf{c}.$$

from conformal field and string theories. Note also that when specializing \mathbf{c} to zero, or equivalently rescaling \mathbf{c} by extra parameter and then letting the parameter degenerate to zero, one recovers the q -deformed Witt algebra. Because of this the Witt algebra is called also, primarily in the physics literature, a centerless Virasoro algebra. In a similar way the q -deformed Witt algebra could be called a centerless q -deformed Virasoro algebra but of course with the word "central" used in terms of the Hom-Lie algebra bracket.

Next, we will link the q -deformed Witt algebras to the framework of deformation theory of Hom-Lie algebras as we have developed. To this end, let $q = 1 + t$. For $n \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$ we have

$$\begin{aligned} \{n\}_q &= \sum_{j=0}^{n-1} q^j = \sum_{j=0}^{n-1} (1+t)^j = \sum_{j=0}^{n-1} \sum_{k=0}^j \binom{j}{k} t^k = \sum_{k=0}^{n-1} \left(\sum_{j=k}^{n-1} \binom{j}{k} \right) t^k \\ q^n &= (1+t)^n = \sum_{k=0}^n \binom{n}{k} t^k, \end{aligned}$$

and thus $\alpha_t(x_n) = (q^n + 1)x_n = (1 + \sum_{k=0}^n \binom{n}{k} t^k)x_n = (1 + \sum_{k=0}^n \binom{n}{k} x_n t^k)$. So, in the decomposition

$$\alpha_t(x) = \sum_{k \geq 0} \alpha_k(x) t^k, \quad \text{where } \alpha_k \in \text{Hom}(V, V)$$

we have $\alpha_0(x_n) = 2x_n$ and $\alpha_k(x_n) = \binom{n}{k} x_n$ for $k > 0$ and $n \in \mathbb{Z}$. In particular, $\alpha_1(x_n) = \binom{n}{1} x_n = nx_n$. For $n, m \in \mathbb{Z}_{\geq 0}$, the relations (6.1) defining the Hom-Lie bracket on generators can be rewritten in terms of t as follows:

$$\begin{aligned} [x_n, x_m]_t &= \sum_{k=0}^{\max(n,m)-1} \left(\left(\sum_{j=k}^{n-1} \binom{j}{k} - \sum_{j=k}^{m-1} \binom{j}{k} \right) x_{n+m} \right) t^k \\ &= \sum_{k \in \mathbb{Z}_{\geq 0}} [x_n, x_m]_k t^k, \end{aligned}$$

where

$$[x_n, x_m]_k = \left(\sum_{j=k}^{n-1} \binom{j}{k} - \sum_{j=k}^{m-1} \binom{j}{k} \right) x_{n+m} \quad \text{for } k \in \mathbb{Z}_{\geq 0}.$$

In particular, for $k = 0$ we have

$$[x_n, x_m]_0 = \left(\sum_{j=0}^{n-1} \binom{j}{0} - \sum_{j=0}^{m-1} \binom{j}{0} \right) x_{n+m} = (n - m)x_{n+m}$$

meaning that the Witt algebra is exactly present in the zero degree term (origin) of the deformation. In the first order term, $k = 1$, we get

$$\begin{aligned} [x_n, x_m]_1 &= \left(\sum_{j=1}^{n-1} \binom{j}{1} - \sum_{j=1}^{m-1} \binom{j}{1} \right) x_{n+m} \\ &= \left(\frac{n(n-1)}{2} - \frac{m(m-1)}{2} \right) x_{n+m} = \frac{(n-m)(n+m-1)}{2} x_{n+m}. \end{aligned}$$

Proposition 6.4. Consider the Witt algebra $W_{\geq 0} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{K}x_n$ defined by the brackets

$$[x_n, x_m]_0 = (n - m)x_{n+m}.$$

The one parameter families given by

$$[x_n, x_m]_t = \sum_{k \in \mathbb{Z}_{\geq 0}} [x_n, x_m]_k t^k \quad \text{and} \quad \alpha_t(x) = \sum_{k \in \mathbb{Z}_{\geq 0}} \alpha_k(x) t^k$$

where

$$\begin{aligned} [x_n, x_m]_k &= \left(\sum_{j=k}^{n-1} \binom{j}{k} - \sum_{j=k}^{m-1} \binom{j}{k} \right) x_{n+m} \quad \text{for } k \in \mathbb{Z}_{\geq 0}, \\ \alpha_0(x_n) &= 2x_n, \quad \alpha_k(x_n) = \binom{n}{k} x_n \quad \text{for } k > 0, n \in \mathbb{Z} \end{aligned}$$

define a Hom-Lie algebra deformation of the Witt algebra $W_{\geq 0}$.

Remark 6.5. Consider $W_{\geq 0} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{K}x_n$. Let $[\cdot, \cdot]_1$ be a skew-symmetric bilinear map on $W_{\geq 0}$ and α_1 be a linear map on $W_{\geq 0}$, defined by

$$[x_n, x_m]_1 = \frac{(n-m)(n+m-1)}{2} x_{n+m}$$

and

$$\alpha_1(x_n) = nx_n.$$

Then the pair $([\cdot, \cdot]_1, \alpha_1)$ is not a 2-Hom-cocycle (with respect to the previous cohomology) of the Witt algebra $W_{\geq 0}$ considered as a Hom-Lie algebra with the linear map defined by $\alpha_0(x_n) = 2x_n$.

One has

$$\begin{aligned} & \circlearrowleft_{p,q,r} [\alpha_0(x_p), [x_q, x_r]_0]_1 + [\alpha_1(x_p), [x_q, x_r]_0]_0 + [\alpha_0(x_p), [x_q, x_r]_1]_0 \\ &= \circlearrowleft_{p,q,r} [2x_p, (q-r)x_{q+r}]_1 + [px_p, (q-r)x_{q+r}]_0 + [2x_p, \frac{(q-r)(q+r-1)}{2}x_{q+r}]_0 \\ &= \circlearrowleft_{p,q,r} 2(q-r)(p-q-r)(p+q+r-1)x_{p+q+r} = 0 \end{aligned}$$

but

$$\circlearrowleft_{p,q,r} [\alpha_0(x_p), [x_q, x_r]_0]_1 + [\alpha_0(x_p), [x_q, x_r]_1]_0 \neq 0$$

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